Linear rigidity of stationary stochastic processes

Alexander I. Bufetov, Yoann Dabrowski, Yanqi Qiu

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Abstract

We consider stationary stochastic processes X_n, $n \in \mathbb{Z}$ such that X_0 lies in the closed linear span of X_n, $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be rigid, that the spectral density vanish at zero and belong to the Zygmund class $\Lambda_*(1)$. We next give sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of Dyson sine-kernels is *not* linearly rigid.

1 Introduction

This paper is devoted to rigidity of stationary determinantal point processes.

Recall that stationary determinantal point processes are strongly chaotic: they have the Kolmogorov property (Lyons [9]) and the Bernoulli property (Lyons and Steif [10]); and they satisfy the Central Limit Theorem (Costin and Lebowitz [2], Soshnikov[13]). On the other hand, Ghosh [5] and Ghosh-Peres [6] proved, for the determinantal point processes such as Dyson sine process and Ginibre point process, that number of particles in a finite window is measurable with respect to the

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completion of the sigma-algebra describing the configurations outside that finite window. Their argument is spectral: they construct, for any small ε, a compactly supported smooth function φ_ε, such that φ_ε equals 1 in a fixed finite window and the linear statistic corresponding to φ_ε has variance smaller than ε.

In the same spirit, we consider general stationary stochastic processes (in broad sense) X_n, $n \in \mathbb{Z}$ such that X_0 lies in the closed linear span of X_n, $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be rigid, that the spectral density vanish at zero and belong to the Zygmund class $\Lambda_*(1)$. We next give sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of Dyson sine-kernels is *not* linearly rigid.

We now turn to more precise statements. Let $X = \{X_n : n \in \mathbb{Z}^d\}$ be a multi-dimensional time stationary stochastic process of real-valued random variables defined on a probability space $(\Omega, \mathbb{P})$. Let $H(X) \subset L^2(\Omega, \mathbb{P})$ denote the closed subspace linearly spanned by $\{X_n : n \in \mathbb{Z}^d\}$ and let $\check{H}_0(X)$ denote the one linearly spanned by $\{X_n : n \in \mathbb{Z}^d \setminus \{0\}\}$.

Definition 1.1. The stochastic process X is said to be linearly rigid if

$$X_0 \in \check{H}_0(X). \tag{1}$$

Let $\mathrm{Conf}(\mathbb{R}^d)$ be the set of configurations on $\mathbb{R}^d$. For a bounded Borel subset $B \subset \mathbb{R}^d$, we denote $N_B : \mathrm{Conf}(\mathbb{R}^d) \to \mathbb{N} \cup \{0\}$ the function defined by

$$N_B(\mathcal{X}) := \text{ the cardinality of } B \cap \mathcal{X}\ .$$

The space $\mathrm{Conf}(\mathbb{R}^d)$ is equipped with the Borel σ-algebra which is the smallest σ-algebra making all N_B's measurable. Recall that a point process with phase space $\mathbb{R}^d$ is, by definition, a Borel probability measure on the space $\mathrm{Conf}(\mathbb{R}^d)$. For the background on point process, the reader is referred to Daley and Vere-Jones' book [3].

Given a stationary point process on $\mathbb{R}^d$ and $\lambda > 0$, we introduce the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ by the formula

$$N_n^{(\lambda)}(\mathcal{X}) := \text{ the cardinality of } \mathcal{X} \cap \big(n\lambda + [-\lambda/2, \lambda/2)^d\big). \tag{2}$$

Definition 1.2. A stationary point process $\mathbb{P}$ on $\mathbb{R}^d$ is called **linearly rigid**, if for any $\lambda > 0$, the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n\in\mathbb{Z}^d}$ is linearly rigid, i.e.,

$$N_0^{(\lambda)} \in \check{H}_0(N^{(\lambda)}).$$

The above definition is motivated by the definition due to Ghosh and Peres of rigidity of point processes on $\mathbb{R}^d$, see [5] and [6]. Given a Borel subset $C \subset \mathbb{R}^d$, we will denote

$$\mathcal{F}_C = \sigma(\{N_B : B \subset C, B \text{ bounded Borel}\})$$

the σ-algebra generated by all random variables of the form N_B where $B \subset C$ ranges over all bounded Borel subsets of C. Let $\mathbb{P}$ be a point process on $\mathbb{R}$, i.e., $\mathbb{P}$ is a Borel probability on $\mathrm{Conf}(\mathbb{R}^d)$, and denote $\mathcal{F}_C^{\mathbb{P}}$ for the $\mathbb{P}$-completion of $\mathcal{F}_C$.

Definition 1.3 (Ghosh [5], Ghosh-Peres [6])**.** A point process $\mathbb{P}$ on $\mathbb{R}^d$ is called **rigid**, if for any bounded Borel set $B \subset \mathbb{R}^d$ with Lebesgue-negligible boundary ∂B, the random variable N_B is $\mathcal{F}^{\mathbb{P}}_{\mathbb{R}^d\setminus B}$-measurable.

Remark 1.1. Of course, in the above definition, it suffices to take Borel sets B of the form $[-\gamma, \gamma)^d$ for $\gamma > 0$, cf. [6].

A linear rigid stationary point process on $\mathbb{R}^d$ is of course rigid in the sense of Ghosh and Peres. Observe that proofs for rigidity in [5], [6] and [1] in fact establish linear rigidity. We would like also to mention a notion of insertion-deletion tolerance studied by Holroyd and Soo in [7], which is in contrast to the notion of rigidity property.

2 The Kolmogorov criterion for linear rigidity

In this note, the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi x\cdot\xi} dx.$$

Denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus. In what follows, we identify $\mathbb{T}^d$ with $[-1/2, 1/2)^d$. The Fourier coefficients of a measure μ on $\mathbb{T}^d$ are given, for any $k \in \mathbb{Z}^d$, by the formula

$$\hat{\mu}(k) = \int_{\mathbb{T}^d} e^{-i2\pi k\cdot\theta} d\mu_X(\theta), \text{ where } k\cdot\theta := k_1\theta_1 + \cdots + k_d\theta_d.$$

Denote by μ_X the spectral measure of X, i.e.,

$$\forall k \in \mathbb{Z}^d, \quad \mathbb{E}(X_0 X_k) = \mathbb{E}(X_n X_{n+k}) = \int_{\mathbb{T}^d} e^{-i2\pi k\cdot\theta} d\mu_X(\theta) = \hat{\mu}_X(k). \tag{3}$$

Recall that we have the following natural isometric isomorphism

$$H(X) \simeq L^2(\mathbb{T}^d, \mu_X), \tag{4}$$

by assigning to $X_n \in H(X)$ the function $\theta \mapsto e^{i2\pi n\cdot\theta} \in L^2(\mathbb{T}^d, \mu_X)$.

Let $\mu_X = \mu_a + \mu_s$ be the Lebesgue decomposition of μ_X with respect to the normalized Lebesgue measure $m(d\theta) = d\theta_1 \cdots d\theta_d$ on $\mathbb{T}^d$, i.e., μ_a is absolutely continuous with respect to m and μ_s is singular to m. Set

$$\omega_X(\theta) := \frac{d\mu_a}{dm}(\theta).$$

Lemma 2.1 (The Kolmogorov Criterion)**.** *We have*

$$\mathrm{dist}(X_0, \check{H}_0(X)) = \left(\int_{\mathbb{T}^d} \omega_X^{-1} dm\right)^{-1/2}.$$

The right side is to be interpreted as zero if $\int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty$.

When the measure μ is assumed to be absolutely continuous with respect to m, Lemma 2.1 is a result of Kolmogorov, see Remark 5.17 in Lyons-Steif [10].

Corollary 2.2. *The stationary stochastic process* $X = (X_n)_{n\in\mathbb{Z}^d}$ *is linearly rigid if and only if*

$$\int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty.$$

Proof of Lemma 2.1. We follow the argument of Lyons-Steif [10]. By the Lebesgue decomposition of μ, we may take a subset $A \subset \mathbb{T}^d$ of full Lebesgue measure $m(A) = 1$, such that $\mu_a(A) = 1$ and $\mu_s(A) = 0$.

Denote

$$L_0 = \overline{\mathrm{span}}^{L^2(\mathbb{T}^d, \mu_X)}[e^{i2\pi n\cdot\theta} : n \neq 0].$$

By the isometric isomorphism (4), it suffices to show that

$$\mathrm{dist}(1, L_0) = \left(\int_{\mathbb{T}^d} \omega_X^{-1} dm\right)^{-1/2}, \tag{5}$$

where 1 is the constant function taking value 1. Write

$$1 = p + h, \text{ such that } p \perp L_0, h \in L_0.$$

Modifying, if necessary, the values of p and h on a μ-negligible subset, we may assume that

$$1 = p(\theta) + h(\theta) \text{ for all } \theta \in \mathbb{T}^d.$$

Since $p \perp L_0$, we have

$$0 = \langle p, e^{i2\pi n\cdot\theta}\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} p(\theta)e^{-i2\pi n\cdot\theta} d\mu(\theta), \text{ for any } n \in \mathbb{Z}^d \setminus 0. \tag{6}$$

Therefore, the complex measure $p \cdot d\mu$ is a multiple of Lebesgue measure, i.e., there exists $\xi \in \mathbb{C}$, such that

$$p \cdot d\mu = \xi dm.$$

It follows that p must vanish almost everywhere with respect to the singular component μ_s of μ, and $p(\theta)\omega_X(\theta) = \xi$ for m-almost every $\theta \in \mathbb{T}^d$. Thus we have

$$\|p\|_{L^2(d\mu)} = \|p\|_{L^2(d\mu_a)}, \tag{7}$$

and

$$h(\theta) = 1 - \xi\omega_X(\theta)^{-1} \text{ for } m\text{-almost every } \theta \in \mathbb{T}^d. \tag{8}$$

Case 1: $\int_{\mathbb{T}^d} \omega_X^{-1} dm < \infty$.

Define a function $f : \mathbb{T}^d \to \mathbb{C}$ by $f = \omega_X^{-1}\chi_A$. Then $f \in L^2(d\mu) \ominus L_0$. Indeed,

$$\|f\|^2_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X^{-2}\chi_A d\mu = \int_{\mathbb{T}^d} \omega_X^{-2} d\mu_a = \int_{\mathbb{T}^d} \omega_X^{-1} dm < \infty.$$

And, for all $n \in \mathbb{Z}^d \setminus 0$,

$$\langle f, e^{i2\pi n\cdot\theta}\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X(\theta)^{-1}\chi_A(\theta)e^{-i2\pi n\cdot\theta} d\mu(\theta) = \int_{\mathbb{T}^d} e^{-i2\pi n\cdot\theta} dm(\theta) = 0.$$

It follows that $f \perp h$, i.e.,

$$0 = \langle h, f\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} h\omega_X^{-1}\chi_A d\mu = \int_{\mathbb{T}^d} h dm.$$

By (8), we get

$$\int_{\mathbb{T}^d}(1-\xi\omega_X^{-1})dm = 0,$$

and hence

$$\xi = (\int_{\mathbb{T}^d}\omega_X^{-1}dm)^{-1}.$$

It follows that

$$\mathrm{dist}(1, L_0)^2 = \|p\|^2_{L^2(d\mu)} = \|p\|^2_{L^2(d\mu_a)} = \xi^2\int_{\mathbb{T}^d}\omega_X^{-2}\omega_X dm = \xi.$$

This shows the desired equality (5).

Case 2: $\int_{\mathbb{T}^d}\omega_X^{-1}dm = \infty$.

We claim that $\xi = 0$. If the claim were verified, then we would get the desired identity in this case

$$\mathrm{dist}(1, L_0) = 0.$$

So let us turn to the proof of the claim. We argue by contradiction. If $\xi \neq 0$, then $p \neq 0$ and

$$\|p\|^2_{L^2(d\mu)} = \|p\|^2_{L^2(d\mu_a)} = \xi^2\|\omega_X^{-1}\|^2_{L^2(d\mu_a)} = \xi^2\int_{\mathbb{T}^d}\omega_X^{-1}dm = \infty.$$

This contradicts the fact that $p \in L^2(d\mu)$. □

Remark 2.1. The same argument shows that, in the case of one-dimensional time, the following assertions are equivalent:

- $\sum_{k=-n}^{n} X_k \in \overline{\mathrm{span}}\{X_j : |j| \geq n+1\}$;
- for any $\alpha_1, \cdots, \alpha_n \in (-1/2, 1/2)\setminus\{0\}$, we have

$$\int_{\mathbb{T}}\frac{\prod_{j=1}^{n}|e^{i2\pi\theta}-e^{i2\pi\alpha_j}|^2|e^{i2\pi\theta}-e^{-i2\pi\alpha_j}|^2}{\omega_X(\theta)}dm(\theta) = \infty.$$

It would be interesting to find a similar characterization for multi-dimensional time as well.

Denote by $\mathrm{Cov}(U,V)$ the covariance between two random variables U and V: $\mathrm{Cov}(U,V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)$.

If $X = (X_n)_{n\in\mathbb{Z}^d}$ is a stochastic process such that

$$\sum_{n\in\mathbb{Z}^d} |\mathrm{Cov}(X_0, X_n)| < \infty, \tag{9}$$

then we may define a continuous function on $\mathbb{T}^d$ by the formula

$$\omega_X(\theta) := \sum_{n\in\mathbb{Z}^d} \mathrm{Cov}(X_0, X_n) e^{i2\pi n\cdot\theta}. \tag{10}$$

Lemma 2.3. *Let $X = (X_n)_{n\in\mathbb{Z}^d}$ be a stationary stochastic process satisfying condition (9). Then we have the following explicit Lebesgue decomposition of μ_X:*

$$\mu_X = (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m, \tag{11}$$

where δ_0 is the Dirac measure on the point $0 \in \mathbb{T}^d$ and ω_X is the function on $\mathbb{T}^d$ defined by (10).

Proof. Note that, under the assumption (9), the function $\omega_X(\theta)$ is well-defined and continuous on $\mathbb{T}^d$. For proving the decomposition (11), it suffices to show that the Fourier coefficients of μ_X coincide with those of $\nu_X := (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m$. But if $n \in \mathbb{Z}^d$, then

$$\hat{\nu}_X(n) = (\mathbb{E}X_0)^2 + \mathrm{Cov}(X_0, X_n) = \mathbb{E}(X_0 X_n) = \hat{\mu}_X(n).$$

The lemma is completely proved. $\square$

3 A sufficient condition for linear rigidity

Theorem 3.1. *Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary stochastic process. If*

$$\sup_{N\geq 1} \left(N \sum_{|n|\geq N} |\mathrm{Cov}(X_0, X_n)| \right) < \infty, \tag{12}$$

and

$$\sum_{n\in\mathbb{Z}} \mathrm{Cov}(X_0, X_n) = 0. \tag{13}$$

Then X is linearly rigid.

Remark 3.1. The condition (12) is a sufficient condition such that the spectral density ω_X is a function in the Zygmund class $\Lambda_*(1)$, see below for definition. The condition (13) implies in particular that ω_X vanishes at the point $0 \in \mathbb{T}$.

We shall apply a result of F. Móricz [12, Thm. 3] on absolutely convergent Fourier series and Zygmund class functions. Recall that a continuous 1-periodic function φ defined on $\mathbb{R}$ is said to be in the *Zygmund class* $\Lambda_*(1)$, if there exists a constant C such that

$$|\varphi(x+h) - 2\varphi(x) + \varphi(x-h)| \leq Ch \tag{14}$$

for all $x \in \mathbb{R}$ and for all $h > 0$.

Theorem 3.2 (Móricz, [12]). *If $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N\geq 1} \left(N \sum_{|n|\geq N} |c_n| \right) < \infty, \tag{15}$$

then the function $\varphi(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{i2\pi n\theta}$ is in the Zygmund class $\Lambda_*(1)$.

Proof of Theorem 3.1. First, in view of (10), our assumption (13) implies

$$\omega_X(0) = 0.$$

Next, by Theorem 3.2, under the assumption (12), we have

$$\omega_X \in \Lambda_*(1).$$

Since all Fourier coefficients of ω_X are real, we have

$$\omega_X(\theta) = \omega_X(-\theta).$$

Consequently, there exists $C > 0$, such that

$$\omega_X(\theta) = \frac{\omega_X(\theta) + \omega_X(-\theta)}{2} = \frac{\omega_X(\theta) + \omega_X(-\theta) - 2\omega_X(0)}{2} \leq C|\theta|,$$

whence

$$\int_{\mathbb{T}} \omega_X^{-1} dm = \infty,$$

and the stochastic process $X = (X_n)_{n\in\mathbb{Z}}$ is linearly rigid by the Kolmogorov criterion. $\square$

4 Applications to stationary determinantal point processes

In this section, we first give a sufficient condition for linear rigidity of stationary determinantal point processes on $\mathbb{R}$ and then give an example of a very simple stationary, but not linearly rigid, determinantal point process on $\mathbb{R}^2$. We briefly recall the main definitions. Let $B \subset \mathbb{R}^d$ be a bounded Borel subset. Let $K_B : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator of convolution with the Fourier transform $\widehat{\chi_B}$ of the indicator function χ_B. In other words, the kernel of K_B is

$$K_B(x, y) = \widehat{\chi_B}(x - y). \tag{16}$$

In particular, if $d = 1$ and $B = (-1/2, 1/2)$, then we find the well-known Dyson sine kernel

$$K_{\text{sine}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

Note that we always have $K_B(x, x) = K_B(0, 0)$.

Denote by $\mathbb{P}_{K_B}$ the determinantal point process induced by K_B. For the background on the determinantal point processes, the reader is referred to [8], [9], [11], [13].

Proposition 4.1. *Let $\mathbb{P}_{K_B}$ be the stationary determinantal point process on $\mathbb{R}^d$ induced by the kernel K_B in (16). For any $\lambda > 0$, denote by $N^{(\lambda)} = (N_n^{(\lambda)})_{n\in\mathbb{Z}^d}$ the stationary stochastic process associated to $\mathbb{P}_{K_B}$ as in (2). Then*

$$\sum_{n\in\mathbb{Z}^d} |\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| < \infty \tag{17}$$

and

$$\sum_{n\in\mathbb{Z}^d} \mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \tag{18}$$

Proof. Fix a number $\lambda > 0$, for simplifying the notation, let us denote $N_n^{(\lambda)}$ by N_n. Denote for any $n \in \mathbb{Z}^d$,

$$Q_n = n\lambda + [-\lambda/2, \lambda/2)^d.$$

By definition of a determinantal point process, we have

$$\mathbb{E}(N_n) = \mathbb{E}(N_0) = \int_{Q_0} K_B(x, x)dx = \lambda^d K_B(0, 0).$$

If $n \neq 0$, we have

$$\mathbb{E}(N_0 N_n) = \iint \chi_{Q_0}(x)\chi_{Q_n}(y) \begin{vmatrix} K_B(x,x) & K_B(x,y) \\ K_B(y,x) & K_B(y,y) \end{vmatrix} dxdy$$
$$= \lambda^{2d} K_B(0,0)^2 - \iint_{Q_0 \times Q_n} |K_B(x,y)|^2 dxdy,$$

whence

$$\mathrm{Cov}(N_0, N_n) = -\iint_{Q_0 \times Q_n} |K_B(x,y)|^2 dxdy. \tag{19}$$

We also have

$$\mathbb{E}(N_0^2) = \mathbb{E}\left[\sum_{x,y \in \mathfrak{X}} \chi_{Q_0}(x)\chi_{Q_0}(y)\right]$$
$$= \mathbb{E}\left[\sum_{x \in \mathfrak{X}} \chi_{Q_0}(x)\right] + \mathbb{E}\left[\sum_{x,y \in \mathfrak{X}, x \neq y} \chi_{Q_0}(x)\chi_{Q_0}(y)\right]$$
$$= \int_{Q_0} K_B(x,x)dx + \iint \chi_{Q_0}(x)\chi_{Q_0}(y) \begin{vmatrix} K_B(x,x) & K_B(x,y) \\ K_B(y,x) & K_B(y,y) \end{vmatrix} dxdy$$
$$= \lambda^d K_B(0,0) + \lambda^{2d} K_B(0,0)^2 - \iint_{Q_0 \times Q_0} |K_B(x,y)|^2 dxdy,$$

whence

$$\mathrm{Cov}(N_0, N_0) = \mathrm{Var}(N_0) = \lambda^d K_B(0,0) - \iint_{Q_0 \times Q_0} |K_B(x,y)|^2 dxdy. \tag{20}$$

Now recall that K_B is an orthogonal projection. Thus we have

$$K_B(0,0) = K_B(x,x) = \int |K_B(x,y)|^2 dy = \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x,y)|^2 dy. \tag{21}$$

The identities (19), (20) and (21) imply that

$$\sum_{n \in \mathbb{Z}^d} \mathrm{Cov}(N_0, N_n) = \lambda^d K_B(0,0) - \int_{Q_0} dx \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x,y)|^2 dy$$
$$= \lambda^d K_B(0,0) - \lambda^d K_B(0,0) = 0.$$

Moreover, the above series converge absolutely. Proposition 4.1 is completely proved. $\square$

Remark 4.1. By Lemma 2.3 and Proposition 4.1, we see that for any stationary determinantal point process induced by a projection, the spectral density of the associated stochastic process $N^{(\lambda)}$ always vanishes at 0.

4.1 Stationary determinantal point processes on $\mathbb{R}$

Theorem 4.2. *Assume that $B \subset \mathbb{R}$ satisfies*

$$\sup_{R>0} \left(R \int_{|\xi| \geq R} |\widehat{\chi_B}(\xi)|^2 d\xi \right) < \infty. \tag{22}$$

Then the stationary determinantal point process $\mathbb{P}_{K_B}$ is linearly rigid.

Proof. By definition of linear rigidity, we need to show that for any $\lambda > 0$, the stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}}$ is linearly rigid. As in the proof of Proposition 4.1, we denote $N_n^{(\lambda)}$ by N_n. By Theorem 3.1, it suffices to show that

$$\sup_{N \geq 1} \left(N \sum_{|n| \geq N} |\mathrm{Cov}(N_0, N_n)| \right) < \infty, \tag{23}$$

and

$$\sum_{n \in \mathbb{Z}} \mathrm{Cov}(N_0, N_n) = 0. \tag{24}$$

By Proposition 4.1, the identity (24) holds in general case. It remains to prove (23). By (19), we have

$$\begin{aligned} \sup_{N \geq 1} \left(N \sum_{|n| \geq N} |\mathrm{Cov}(N_0, N_n)| \right) &= \sup_{N \geq 1} N \iint_{\bigcup\limits_{|n| \geq N} Q_n} |\widehat{\chi}_B(x-y)|^2 dx dy \\ &\leq \sup_{N \geq 1} \lambda N \int_{|\xi| \geq (N-1)\lambda} |\widehat{\chi}_B(\xi)|^2 d\xi < \infty \end{aligned}$$

where in the last inequality, we used our assumption (22). Theorem 4.2 is proved completely. $\square$

Remark 4.2. When B is a finite union of finite intervals on the real line, the rigidity of the stationary determinantal point process $\mathbb{P}_{K_B}$ is due to Ghosh [5].

4.2 Tensor product of sine kernels

In higher dimension, the situation becomes quite different. Let

$$S = I \times I = (-1/2, 1/2) \times (-1/2, 1/2) \subset \mathbb{R}^2.$$

Then the associate kernel K_S has a tensor form: $K_S = K_{\text{sine}} \otimes K_{\text{sine}}$, that is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$K_S(x, y) = K_{\text{sine}}(x_1, y_1) K_{\text{sine}}(x_2, y_2) = \frac{\sin(\pi(x_1 - y_1))}{\pi(x_1 - y_1)} \frac{\sin(\pi(x_2 - y_2))}{\pi(x_2 - y_2)}.$$

Proposition 4.3. *The determinantal point process $\mathbb{P}_{K_S}$ is not linearly rigid. More precisely, let $N^{(1)} = (N_n^{(1)})_{n \in \mathbb{Z}^2}$ be the stationary stochastic process given as in Definition 1.2, then*

$$N_0^{(1)} \notin \check{H}_0(N^{(1)}).$$

To prove the above result, we need to introduce some extra notation. First, we define the multiple Zygmund class Λ_* as follows. A continuous function $\varphi(x, y)$ periodic in each variable with period 1 is said to be in the multiple Zygmund class $\Lambda_*(1, 1)$ if for the double difference difference operator $\Delta_{2,2}$ of second order in each variable, applied to φ, there exists a constant $C > 0$, such that for all $x = (x_1, x_2) \in (-1/2, 1/2) \times (-1/2, 1/2)$ and $h_1, h_2 > 0$, we have

$$|\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2)| \leq C h_1 h_2, \tag{25}$$

where

$$\begin{aligned}\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2) &:= \varphi(x_1 + h_1, x_2 + h_2) + \varphi(x_1 - h_1, x_2 + h_2) \\ &+ \varphi(x_1 + h_1, x_2 - h_2) + \varphi(x_1 - h_1, x_2 - h_2) - 2\varphi(x_1 + h_1, x_2) \\ &- 2\varphi(x_1 - h_1, x_2) - 2\varphi(x_1, x_2 + h_2) - 2\varphi(x_1, x_2 - h_2) + 4\varphi(x_1, x_2).\end{aligned}$$

The following result is due to Fülöp and Móricz [4, Thm 2.1 and Rem. 2.3]

Theorem 4.4 (Fülöp-Móricz)**.** *If $\{c_{jk}\}_{j,k \in \mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N \geq 1, M \geq 1} \left(MN \sum_{|j| \geq N, |k| \geq M} |c_{jk}| \right) < \infty, \tag{26}$$

then the function

$$\varphi(\theta_1, \theta_2) = \sum_{j,k \in \mathbb{Z}} c_{jk} e^{i2\pi(j\theta_1 + k\theta_2)}$$

is in the Zygmund class $\Lambda_*(1, 1)$.

Let us turn to the study of the density function $\omega_{N^{(1)}}$.

Lemma 4.5. *There exists* $c > 0$*, such that for any* $\theta_1, \theta_2 \in (-1/2, 1/2)$*, we have*

$$\omega_{N^{(1)}}(\theta_1, \theta_2) \geq c(|\theta_1| + |\theta_2|).$$

Proof. To make notation lighter, in this proof we simply write ω for $\omega_{N^{(1)}}$.

Denote $S_n = S \times (n + S)$ where $n + S := (-1/2 + n_1, 1/2 + n_1) \times (-1/2 + n_2, 1/2 + n_2)$. By the same argument as in the proof of Theorem 4.2, we obtain that for any $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus 0$,

$$\widehat{\omega}(n) = -\int_{S_n} |K_S(x, y)|^2 dxdy,$$

and

$$\widehat{\omega}(0) = K_S(0, 0) - \int_{S_0} |K_S(x, y)|^2 dxdy.$$

The following properties can be easily checked.

- $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0.$
- $\widehat{\omega}(\varepsilon_1 n_1, \varepsilon_2 n_2) = \widehat{\omega}(n_1, n_2)$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.
- there exist $c, C > 0$, such that

$$\frac{c}{(1 + n_1^2)(1 + n_2^2)} \leq |\widehat{\omega}(n_1, n_2)| \leq \frac{C}{(1 + n_1^2)(1 + n_2^2)}.$$

For instance, $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0$ follows from Proposition 4.1. These properties combined with Theorem 4.4 yield that

- $\omega(0, 0) = 0.$
- $\omega(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2) = \omega(\theta_1, \theta_2)$ for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\theta_1, \theta_2 \in (-1/2, 1/2)$.
- the function $\omega(\theta_1, \theta_2)$ is in the multiple Zygmund class $\Lambda_*(1, 1)$.

Hence there exists $C > 0$, such that

$$|\omega(\theta_1, \theta_2) - \omega(\theta_1, 0) - \omega(0, \theta_2)| \leq C|\theta_1 \theta_2|. \tag{27}$$

Lemma 4.6. *There exists $c > 0$, such that*

$$\omega(\theta_1, 0) \geq c|\theta_1| \textit{ and } \omega(0, \theta_2) \geq c|\theta_2|. \tag{28}$$

Let us postpone the proof of Lemma 4.6 and proceed to the proof of Lemma 4.5. The inequalities (27) and (28) imply that

$$\omega(\theta_1, \theta_2) \geq c(|\theta_1| + |\theta_2|) - C|\theta_1\theta_2|.$$

To prove the lower bound of type as in the lemma, it suffices to prove it when $|\theta_1|$ and $|\theta_2|$ are small enough, for instance, $2C|\theta_1| \leq c$, then we have

$$\omega(\theta_1, \theta_2) \geq \frac{c}{2}(|\theta_1| + |\theta_2|).$$

□

Now let us turn to the

Proof of Lemma 4.6. By symmetry, it suffices to prove that there exists $c > 0$, such that $\omega(\theta_1, 0) \geq |\theta_1|$. To this end, let us denote $\omega_1(\theta_1) := \omega(\theta_1, 0)$. Then $\omega_1(0) = 0$ and there exists $c > 0$ such that if $k \neq 0$, then

$$\widehat{\omega}_1(k) < 0 \text{ and } |\widehat{\omega}_1(k)| \geq c/(1 + k^2),$$

Indeed, we have

$$\omega_1(\theta_1) = \sum_{k\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \widehat{\omega}(k, n_2) e^{i2\pi k\theta_1},$$

if $k \neq 0$, then $\widehat{\omega}(k, n_2) < 0$ and hence

$$|\widehat{\omega}_1(k)| = \sum_{n_2\in\mathbb{Z}} |\widehat{\omega}(k, n_2)| \geq \sum_{n_2\in\mathbb{Z}} \frac{c}{(1 + n_2^2)(1 + k^2)} \geq \frac{c'}{1 + k^2}.$$

Note also that $\omega_1(0) = \omega(0, 0) = 0$, hence

$$\sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k) = 0.$$

It follows that

$$\begin{aligned}\omega_1(\theta_1) &= \sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k) e^{i2\pi k\theta_1} = \sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k)(\frac{e^{i2\pi k\theta_1}+e^{-i2\pi k\theta_1}}{2}-1)\\ &= \sum_{k\in\mathbb{Z},k\neq 0} -\widehat{\omega}_1(k)(1-\cos(2\pi k\theta_1)) = \sum_{k\in\mathbb{Z},k\neq 0} |\widehat{\omega}_1(k)|(1-\cos(2\pi k\theta_1))\\ &\geq c'' \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2}(1-\cos(2\pi(2j-1)\theta_1)).\end{aligned}$$

Combining with the classical formulae

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8},$$

$$|\alpha| = \frac{1}{4} - \frac{2}{\pi^2}\sum_{j=1}^{\infty} \frac{\cos(2(2j-1)\pi\alpha)}{(2j-1)^2}, \text{ for } \alpha \in (-1/2, 1/2);$$

we obtain that

$$\omega_1(\theta_1) \geq c''\frac{\pi^2}{2}|\theta_1|.$$

□

Proof of Proposition 4.3. By Lemma 2.1, it suffices to show that

$$\int_{\mathbb{T}^2} \omega_{N^{(1)}}^{-1} dm < \infty. \tag{29}$$

By Lemma 4.5, the inequality (29) follows from the following elementary inequality

$$\int_{|\theta_1|<1/2, |\theta_2|<1/2} \frac{1}{|\theta_1|+|\theta_2|} d\theta_1 d\theta_2 < \infty.$$

□

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References

[1] Alexander I. Bufetov. Rigidity of Determinantal Point Processes with the Airy, the Bessel and the Gamma Kernel. *arXiv:1506.07581, Jun 2015*.

[2] Ovidiu Costin and Joel L. Lebowitz. Gaussian fluctuation in random matrices. *Phys. Rev. Lett.*, 75(1):69–72, 1995.

[3] Daryl J. Daley and David Vere-Jones. *An introduction to the theory of point processes. vol. I. , Elementary theory and methods*. Probability and its applications. Springer, New York, Berlin, Paris, 2003.

[4] Vanda Fülöp and Ferenc Móricz. Absolutely convergent multiple Fourier series and multiplicative Zygmund classes of functions. *Analysis*, 28(3/2008):345–354, 2008.

[5] Subhroshekhar Ghosh. Determinantal processes and completeness of random exponentials: the critical case. *Probability Theory and Related Fields*, pages 1–23, 2014.

[6] Subhroshekhar Ghosh and Yuval Peres. Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues,. *arXiv:1211.3506*.

[7] Alexander E. Holroyd and Terry Soo. Insertion and deletion tolerance of point processes. *Electron. J. Probab.*, 18:no. 74, 24, 2013.

[8] J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág. Determinantal processes and independence. *Probab. Surv.*, 3:206–229, 2006.

[9] Russell Lyons. Determinantal probability measures. *Publ. Math. Inst. Hautes Études Sci.*, (98):167–212, 2003.

[10] Russell Lyons and Jeffrey E. Steif. Stationary determinantal processes: phase multiplicity, Bernoullicity, entropy, and domination. *Duke Math. J.*, 120(3):515–575, 2003.

[11] Odile Macchi. The coincidence approach to stochastic point processes. *Advances in Appl. Probability*, 7:83–122, 1975.

[12] Ferenc Móricz. Absolutely convergent Fourier series and function classes. *J. Math. Anal. Appl.*, 324(2):1168–1177, 2006.

[13] Alexander Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55(5(335)):107–160, 2000.